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TRIGONAL GORENSTEIN CURVES AND SPECIAL LINEAR SYSTEMS

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ABSTRACT

Let Y be a Gorenstein trigonal curve with $g := p_a(Y) \geq 0$. Here we study the theory of special linear systems on Y, extending the classical case of a smooth Y given by Maroni in 1946. As in the classical case, to study it we use the minimal degree surface scroll containing the canonical model of Y. The answer is different if the degree 3 pencil on Y is associated to a line bundle or not. We also give the easier case of special linear series on hyperelliptic curves. The unique hyperelliptic curve of genus g which is not Gorenstein has no special spanned line bundle.

1. Introduction

The main aim of this paper is the extension to singular Gorenstein curves of the theory of special linear systems on trigonal curves. The classical case of a smooth curve is due to Maroni (see [Ma] or [MS], Prop. 1). To have a good picture of linear series on a singular curve Y, it is essential to know even the linear series associated to rank 1 torsion free sheaves which are not locally free. The main step is the classification of all such rank 1 torsion free sheaves, A, with $h^1(Y, A) \neq 0$ and which are spanned. If Y is Gorenstein, the associated linear systems are the so-called free linear systems introduced in [H] and [C], but we will not use their beautiful theory, just working always with spanned sheaves. Then to any such spanned A one can "add base points" and obtain another rank 1 torsion free sheaf B with $A \subset B$ and $h^0(Y, A) = h^0(Y, B)$. Vice versa, if we start with B, the sheaf A is the subsheaf of B spanned by $H^0(Y, B)$. If Y is singular, A may be not locally free even if B is a line bundle. Furthermore, for any fixed A and any fixed integer $x \geq 2$ the set of all such B's with $\deg(B) = \deg(A) + x$ may be reducible.

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For instance, if Y has just one ordinary node, P, the set of all effective degree 2 divisors on Y is the disjoint union of two algebraic sets: the two-dimensional variety, S', of all degree two Cartier divisors and the one-dimensional variety, S'', of all divisors $P+Q$ with $Q \in Y_{\mathrm{reg}}$. Take $x = 2$; we start with A which is not locally free at P. If we add an element of S' we obtain a non-locally free sheaf, while if we add an element of S'' we obtain a line bundle (see Remark 2.12); since to be locally free is an open condition, we obtain in this way quite often reducible $W_d^r(Y)$'s. For curves with only planar singularities one can use [BGS] to have a bound for the dimension of this garbage and hence to have information concerning $\dim(W_d^r(Y))$.

First, we study special linear systems on singular hyperelliptic curves, using their classification proven in [EKS], Appendix with J. Harris. Here the situation is very different for Gorenstein hyperlliptic curves (see Proposition 2.3) and for the unique non-Gorenstein one, $T(g)$ (see Theorem 2.4 and Remark 2.5). In particular $T(g)$ has no special spanned line bundle (except of course $\mathbf{O}_{T(g)}$). Then we study Gorenstein trigonal curves using the existence of a minimal degree ruled surface $S \subset \mathbf{P}^{g-1}$ containing the canonical model of Y. By [RS], Th. 3.6, S is the cone over a rational normal curve of $\mathbf{P}^{g-2}$ if and only if the degree 3 spanned torsion free sheaf, L, on Y is not locally free. If this is the case, the theory of special divisors on Y is very simple (see 2.7, 2.8 and 2.9). If L is locally free we show that the picture is exactly as in the smooth case for spanned line bundles of degree at most $g-1$ (see Theorem 2.10 and Corollary 2.11). Then we consider a spanned torsion free sheaf A with $\deg(A) \leq g-1$ which is not locally free on a set $\mathrm{Sing}(A) \neq \emptyset$. If Y has only ordinary nodes or ordinary cusps at each point of $\mathrm{Sing}(A)$, we show that the picture is described by the spanned special line bundles on the partial normalization, C, of Y in which we normalize Y only at each point of $\mathrm{Sing}(A)$ (see Theorem 2.14 and Corollary 2.16). We remark that the Maroni invariant of C depends on the Maroni invariant of Y and on the position of the set $\mathrm{Sing}(A) \subset S$ in the sense of Definition 2.12.

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2. The results

We work over an algebraically closed field $\mathbf{K}$. We will always use the following notation. Y is an integral projective curve, $g := p_a(Y)$ and $\pi\colon X \to Y$ is the normalization. Let A be a rank 1 torsion free sheaf A on Y. Set $\mathrm{Sing}(A) := \{P \in Y\colon A \text{ is not locally free at } P\}$. Hence $\mathrm{Sing}(A) \subseteq \mathrm{Sing}(Y)$. The integer

$\deg(A)$ is defined by the Riemann–Roch type formula $\chi(A) = \deg(A) + \chi(\mathbf{O}_Y)$. By the duality for one-dimensional Cohen–Macaulay schemes ([AK]), we have $h^1(Y, A) = h^0(Y, \mathrm{Hom}(A, \omega_Y))$. We have $\deg(\mathrm{Hom}(A, \omega_Y)) = 2g-2-\deg(A)$ even if Y is not Gorenstein ([Co], part 2) of Prop. 3.1.6), but we need this formula only for a Gorenstein curve. By localy duality we have $\mathrm{Hom}(\mathrm{Hom}(A, \omega_Y), \omega_Y) \cong A$. If $h^0(Y, A) \geq 2$, we will write $|A|$ for the associated complete linear system; however, we will use the notation $|A|$ only when A is spanned; if Y is Gorenstein, the sheaf A is spanned if and only if the linear system $|A|$ is free in the sense of [C]. We want to study the rank 1 torsion free sheaves L on Y with $h^1(Y, L) \geq 2$ and $h^0(Y, L) \geq 2$. Since $\deg(\mathrm{Hom}(L, \omega_Y)) = 2g - 2 - \deg(L)$, either $\deg(L) \leq g - 1$ or $\deg(\mathrm{Hom}(L, \omega_Y)) \leq g - 1$. Hence it is harmless to assume $\deg(L) \leq g - 1$. Since the subsheaf, L', of L spanned by $H^0(Y, L)$ has $h^0(Y, L') = h^0(Y, L) \geq 2$, $h^1(Y, L') \geq h^1(Y, L) \geq 2$ and $\deg(L') \leq \deg(L)$, we will study only the spanned special rank 1 torsion free sheaves with degree at most $g - 1$; notice that even if L is locally free, L' may be not locally free and hence we cannot avoid to study non-locally free sheaves even if we are interested only in special line bundles.

LEMMA 2.1: *Let Y be an integral projective curve and $\pi\colon X \to Y$ its normalization. Let L be a rank 1 torsion free sheaf on Y. The natural map $u\colon H^0(Y, L) \to H^0(X, \pi^*(L)/\mathrm{Tors}(\pi^*(L)))$ is injective.*

Proof: Set $z := h^0(Y, L)$. We may assume $z > 0$. It is sufficient to prove that for $z-1$ general points P_i, $1 \leq i \leq z-1$, of Y, there is $\sigma \in H^0(X, L)$ with $\sigma(P_i) = 0$ for every i and $\sigma(u) \neq 0$. Since $\mathrm{rank}(L) = 1$ and $z = h^0(Y, L)$, for a general $P \in Y$ there is $\sigma \in H^0(Y, L)$ with $\sigma(P_i) = 0$ for $i \leq z - 1$, $\sigma(P) \neq 0$. Since $\pi|\pi^{-1}(Y_{\mathrm{reg}})\colon \pi^{-1}(Y_{\mathrm{reg}}) \to Y_{\mathrm{reg}}$ is an isomorphism and $P \in Y_{\mathrm{reg}}$, it is obvious that $u(\sigma)(\pi^{-1}(P)) \neq 0$ and hence $u(\sigma) \neq 0$.

Remark 2.2: For every integer $g \geq 2$ there is a complete classification of all pairs (Y, L) such that Y is an integral projective curve with $p_a(Y) = g$ and L is a rank 1 torsion free sheaf on Y with $h^0(Y, L) \geq 2$ and $\deg(L) = 2$ ([EKS], Th. A of the Appendix with J. Harris). Every such L is spanned and has $h^0(Y, L) = 2$. For each Y the sheaf L is unique. The sheaf L is locally free if Y is Gorenstein and in this case $\omega_Y \cong L^{\otimes(g-1)}$. There is a unique hyperelliptic curve (call it $T(g)$) which is not Gorenstein. The curve $T(g)$ is rational and it has a unique singular point; call it P; P is unibranch; call $O \in \mathbf{P}^1$ the unique point with $\pi(O) = P$. The conductor of $\mathbf{O}_{T(g),P}$ in $\mathbf{O}_{\mathbf{P}^1,O}$ is the maximal ideal, $\mathbf{m}$, of the local ring $\mathbf{O}_{\mathbf{P}^1,O}$; if t is a generator of $\mathbf{m}$, $\mathbf{O}_{T(g),P}$ is the subring of $\mathbf{O}_{\mathbf{P}^1,O}$ generated by 1 and the powers t^x with $x \geq g+1$; for $g = 1$ we would obtain an ordinary cusp. For every integer r with $1 \le r \le g-1$ there is a unique rank 1 torsion free sheaf, $T(g,r)$, on $T(g)$, with $\deg(T(g,r)) = 2r$ and $h^0(T(g),T(g,r)) = r+1$ ([EKS], Th. A of the Appendix with J. Harris); we have $T(g,r) = \pi_*(\mathbf{O}_{\mathbf{P}^1}(r))$ and this definition shows immediately that $h^0(T(g),T(g,r)) = h^0(\mathbf{P}^1,\mathbf{O}_{\mathbf{P}^1}(r)) = r+1$ and that the function "degree" behaves badly under push-forwards. Since every proper subsheaf of $T(g,r)$ has smaller degree and it is special, every proper subsheaf, F, of $T(g,r)$ has $h^0(T(g),F) \le r$ by the weak part of Clifford's theorem proved in [EKS], Th. A of the Appendix with J. Harris. Thus $T(g,r)$ is spanned.

PROPOSITION 2.3: *Leg Y be an integral projective Gorenstein hyperelliptic curve with $g := p_a(Y) \ge 2$. Let $L \in \operatorname{Pic}^2(Y)$ be the hyperelliptic line bundle. Let A be a rank 1 spanned torsion free sheaf on Y with $h^1(Y,A) \ne 0$. Then A is locally free and there exists an integer r with $1 \le r \le g-1$ such that $A \cong L^{\otimes r}$, $\deg(A) = 2r$ and $h^0(Y,A) = r+1$.*

Proof: Let $f\colon Y \to \mathbf{P}^1$ be the degree 2 morphism induced by L. Hence f induces an involution, σ, on Y_{reg}. Since A is spanned and $A|Y_{\text{reg}}$ is locally free, the pair $(Y, H^0(Y,A))$ induces a morphism $\phi\colon Y_{\text{reg}} \to \mathbf{P}^r$, $r := h^0(Y,A)-1$. By the duality for Cohen–Macaulay schemes ([AK]) we have $h^0(Y,\operatorname{Hom}(A,\omega_Y)) = h^1(Y,A) \ne 0$. Hence A may be seen as a subsheaf of ω_Y. Since f is the morphism induced by $H^0(Y,\omega_Y)$ and A is a subsheaf of ω_Y, for every $P \in Y_{\text{reg}}$ we have $\phi(P) = \phi(\sigma(P))$, i.e. ϕ factors through $f|Y_{\text{reg}}$. Hence for general points $P_1,\dots,P_r$ of Y_{reg} we have $h^0(Y, A(-P_1-\sigma(P_1)-\cdots-P_r-\sigma(P_r))) \ne 0$, i.e. $h^0(Y,\operatorname{Hom}(L^{\otimes r},A)) \ne 0$. Since $h^0(Y,L^{\otimes r}) = h^0(Y,A)$, A is spanned and every non-zero map $L^{\otimes r} \to A$ is injective, we obtain $L^{\otimes r} \cong A$.

Now we will check that the proof of Theorem A of [EKS], Appendix with J. Harris, gives the following complete description of the special linear systems on the rational hyperelliptic curve $T(g)$.

THEOREM 2.4: *Let A be a spanned rank 1 torsion free sheaf on $T(g)$ with $h^1(T(g),A) \ne 0$. Then there exists a unique integer r with $1 \le r \le g-1$ such that $A \cong T(g,r)$.*

Proof: Set

$$B := \pi^*(A)/\operatorname{Tors}(\pi^*(A)), \quad r := \deg(B), \quad D := \pi^*(\omega_Y)/\operatorname{Tors}(\pi^*(\omega_Y)).$$

Hence B and D are spanned line bundles on $X \cong \mathbf{P}^1$. By Lemma 2.1 the natural map $u\colon H^0(Y,A) \to H^0(X,B)$ is injective. It was checked in [EKS], p. 538, first line of Case 2, that $\deg(D) = g-1$ and that the natural map

$u'\colon H^0(Y,\omega_Y)\to H^0(X,D)$ is an isomorphism. By the duality for locally Cohen–Macaulay schemes and the assumption $h^1(Y,A)\neq 0$, A is a subsheaf of ω_Y. Furthermore, B is a subsheaf of D because there is a generically injective map $\pi^*(A)\to\pi^*(\omega_Y)$. Hence the inclusion u is an isomorphism, i.e. $h^0(Y,A)=r+1$. There is a natural generically injective map $A\to\pi_*\pi^*(A)$ and hence a generically injective map $A\to\pi_*(B)$. Since $T(g,r)\cong\pi_*(B)$ (Remark 2.2) and A is torsion-free, we have an inclusion $A\to T(g,r)$. Since $h^0(Y,A)=h^0(Y,T(g,r))$ and $T(g,r)$ is spanned (Remark 2.2), we have $A\cong T(g,r)$.

Remark 2.5: By Theorem 2.4 there is no spanned special line bundle on $T(g)$ (except $\mathbf{O}_{T(g)}$).

LEMMA 2.6: *Let Y be an integral non-hyperelliptic Gorenstein curve with $g:=p_a(Y)\geq 5$. Assume that Y has two rank 1 torsion free sheaves R, L with $\deg(R)=\deg(L)=3$, $h^0(Y,R)\geq 2$ and $h^0(Y,L)\geq 2$. Then $R\cong L$, $h^0(Y,L)=2$ and L is spanned.*

Proof: Since Y is not hyperelliptic and $\deg(R)=\deg(L)=3$, we have $h^0(Y,R)=h^0(Y,R)=2$ and both R and L are spanned. Since Y is not hyperelliptic, its canonical map is an embedding ([Ro], Th. 15). We will see Y as a linearly normal curve of degree $2g-2$ in $\mathbf{P}^{g-1}$. Take any degree 3 effective divisor D associated to R or L. Since $h^0(Y,R)=h^1(Y,R)\neq 0$, D spans a line of $\mathbf{P}^{g-1}$. As in the classical case we may associate to R (resp. L) a degree $g-2$ surface S_R (resp. S_L) with $Y\subset S_R\subset\mathbf{P}^{g-1}$ (resp. $Y\subset S_L\subset\mathbf{P}^{g-1}$) which is either a cone over a rational normal curve of $\mathbf{P}^{g-2}$ or a smooth rational curve, the first case occurring if and only if R (resp. L) is not locally free ([RS]). Furthermore, S_R and S_L are set-theoretically cut out by the quadrics containing Y; indeed, by [RS] the proof of [AM] for the case Y smooth works for Gorenstein curves; alternatively, one could use [Sc], Th. 3.1, to check this assertion and that $\deg(S_R)=\deg(S_L)=g-2$. Hence $S_R=S_L$. Since every line of S_R (resp. S_L) is spanned by its scheme-theoretic intersection with Y which is a degree 3 divisor of the pencil R (resp. L) we have $R\cong L$.

Example 2.7: Fix integers g, r with $g\geq 3$ and $r\geq 3$. Let C be a Gorenstein hyperelliptic curve with $p_a(C)=g-1$. Let $R\in\mathrm{Pic}^2(Y)$ be the hyperelliptic pencil. For every integer $i\geq 1$ set $B_i:=R^{\otimes i}$ and let $\gamma_i\colon C\to\mathbf{P}^i$ be the morphism induced by the pair $(B_i,H^0(C,B_i))$. Thus γ_i is obtained composing γ_1 with the degree i Veronese embedding of $\mathbf{P}^1$ as rational normal curve of $\mathbf{P}^i$. Fix a point, Q, of the secant variety of the rational normal curve $\gamma_r(C)$ but $P\notin\gamma_r(C)$; we allow the case in which Q is on the tangent developable of $\gamma_r(C)$. Consider the projection $u\colon \mathbf{P}^r \setminus \{Q\} \to \mathbf{P}^{r-1}$. Since $r \geq 3$, $u|\gamma_r(C)$ is birational. Thus $\deg(u(\gamma_r(C))) = r$. By the choice of Q the rational curve $\gamma_r(C)$ is singular. We see easily in arbitrary characteristic that $p_a(u(\gamma_r(C))) \leq 1$. Thus $u(\gamma_r(C))$ is a rational curve with an ordinary node or an ordinary cusp. The morphism $\gamma_r \circ u\colon C \to \mathbf{P}^{r-1}$ induces a subspace, W_r, of $H^0(C, B_r)$ with $\dim(W_r) = r$ and W_r spanning B_r. First assume that Q is not in the tangent developable of $\gamma_r(C)$, i.e. assume the existence of $Q', Q'' \in \gamma_r(C)$, with $Q' \neq Q''$ and Q contained in the line $\langle\{Q', Q''\}\rangle$. Take any $P' \in \gamma_1^{-1}(Q')$ and $P' \in \gamma_1^{-1}(Q'')$ and let Y be the unique genus g curve obtained from Y gluing the points P' and P''; if P' and P'' are smooth points of C, then Y is Gorenstein with an ordinary node at the image of P' and P''. The morphism induced by the pair (B_r, W_r) factors through Y and defines a degree $2r$ morphism $v\colon Y \to \mathbf{P}^{r-1}$. Set $A := v^*(\mathbf{O}_{\mathbf{P}^{r-1}}(1))$. Call $w\colon C \to Y$ the induced morphism with $u = w \circ v$. Thus A is a spanned degree $2r$ line bundle on Y with $h^0(Y, A) \geq r$. Since $g - 1 \geq 2$, the hyperelliptic pencil of C is unique and this shows that Y is not hyperelliptic. Hence by Clifford's theorem ([EKS], Th. A of the Appendix with J. Harris) we have $h^0(Y, A) = r$. We have $h^0(Y, w_*(R)) = 2$. Since $p_a(Y) = p_a(C) + 1$, the proof of [EKS], Lemma 1 of the Appendix with J. Harris, we have $2 \leq \deg(w_*(R)) \leq 3$. Since Y is not hyperelliptic, $w_*(R)$ is a degree 3 torsion free sheaf on Y ([EKS], Th. A of the Appendix with J. Harris). Thus Y is trigonal. Vice versa, given any P', P'' on C_{reg} with $\gamma_1(P') \neq \gamma_1(P'')$, take any Q on the line $\langle\{\gamma_r(P'), \gamma_r(P'')\}\rangle$ and apply the previous construction; we obtain a trigonal curve with C as partial normalization, with a new ordinary node and with a spanned $A \in \mathrm{Pic}^{2r}(Y)$ with $h^0(Y, A) = r$. Now assume that Q is on a tangent line of $\gamma_r(C)$, say of the point O. If there is $P \in C_{\mathrm{reg}}$ with $\gamma_1(P) = O$ and P not a ramification point of γ_1, then we obtain a Gorenstein curve Y with C as paretial normalization, $p_a(Y) = g$, an ordinary cusp as additional singular point and with a spanned $A \in \mathrm{Pic}^{2r}(Y)$ with $h^0(Y, A) = r$. The remaining cases are more complicated but in principle understandable, because C has only planar singularities with multiplicity 2, i.e. (at least in characteristic 0) only singularities of type A_k, i.e. only tacnodes and, perhaps non-ordinary, planar cusps. We stress that in this case we obtain Y as a double covering $f\colon Y \to E$ of a rational curve with an ordinary node or an ordinary cusp, i.e. of a rational curve E with $p_a(E) = 1$. For every integer $i \geq 2$ every $T \in \mathrm{Pic}^i(E)$ is spanned and $f^*(T)$ gives a spanned line bundle on Y with $\deg(T) = 2r$ and $h^0(Y, f^*(T)) \geq r$. In this way from one example of a sheaf on the curve Y we find examples for all integers $r \geq 2$.

THEOREM 2.8: *Let Y be an integral non-hyperelliptic Gorenstein curve with $g :=$*

$p_a(Y) \geq 5$. Assume that Y has a degree 3 free pencil $|L|$ which is not base point free, i.e. such that the associated spanned torsion free sheaf L is not a line bundle. The pencil $|L|$ is unique and it has a unique base point, P, i.e. $\mathrm{Sing}(L) = \{P\}$. *Assume Y of multiplicity 2 at P. There are an integral hyperelliptic Gorenstein curve C with $p_a(C) = g-1$ and a birational morphism $w: C \to Y$ such that $w|w^{-1}(Y\setminus\{P\})$: $w^{-1}(Y\setminus\{P\}) \to Y\setminus\{P\}$ is an isomorphism. Let $R \in \mathrm{Pic}^2(C)$ be the hyperelliptic pencil. Then for all integers r with $1 \leq r \leq g-2$ the rank 1 torsion free sheaf $w_*(R^{\otimes r})$ on Y is spanned and has* $\deg(w_*(R^{\otimes r})) = 2r+1$ *and $h^0(Y, w_*(R^{\otimes r})) = r+1$. Let A be a spanned rank 1 torsion free sheaf on Y with* $\deg(A) \leq g-1$. *If A is not locally free we have $A \cong w_*(R^{\otimes r})$ with $r := h^0(Y,A)-1$. If A is locally free, then* $\deg(A) = 2(h^0(Y,A))$ *and the pair (Y,A) arises from C from the construction of Example 2.7.*

Proof: By [RS], Th. 3.6, $|L|$ is the unique degree 3 pencil on Y. Since Y is Gorenstein and non-hyperelliptic, the canonical map, j, of Y is an embedding ([Ro], Th. 15). By assumption $\mathrm{Sing}(L) \neq \emptyset$. By [RS], Th. 3.6, the canonical curve $j(Y) \subset \mathbf{P}^{g-1}$ is contained in the cone, S, over a rational normal curve, D, of $\mathbf{P}^{g-2}$. Projecting from the vertex, P, of S we see that $P \in j(Y)$ and that $j(Y)$ has multiplicity 2 at P. Furthermore, $\{P\} = \mathrm{Sing}(L)$. Let $v: S' \to S$ be the blowing-up of the vertex of the cone S. Hence S' is isomorphic to the Hirzebruch surface F_{g-2}. We take as base of $\mathrm{Pic}(S') \cong \mathbf{Z}^{\otimes 2}$ the curve, h, contracted by v and a fiber, f, of the ruling of S'. Hence $v^*(\mathbf{O}_S(1)) \cong \mathbf{O}_{S'}(h+(g-2)f)$. Let $C \subset S'$ be the strict transform of $j(Y)$ in S' and $w: C \to j(Y) \cong Y$ the induced map. Since $\deg(h(Y)) = 2g-2$ and $C \cdot f = 2$, we have $\mathbf{O}_{S'}(C) \cong \mathbf{O}_{S'}(2h+(2g-2)f)$. By the adjunction formula we obtain $p_a(C) = g-1$. We do not claim that C is smooth along $w^{-1}(P)$, i.e. we do not claim that $h(Y)$ has an ordinary node or an ordinary cusp at P. However, since S' is smooth and $C \cdot f = 2$, C is a Gorenstein hyperelliptic curve with $p_a(C) = g-1 \geq 4$. Call $R \in \mathrm{Pic}^2(C)$ the hyperelliptic pencil. Set $B := w^*(A)/\mathrm{Tors}(w^*(A))$. Hence B is a rank 1 spanned torsion free sheaf on C. The proof of Lemma 2.1 gives $h^0(C,B) \geq 2$, i.e. B is not trivial. Since $p_a(Y) - p_a(C) = 1$, the proof of [EKS], Lemma 1 of the Appendix with J. Harris, gives $\deg(A) - 1 \leq \deg(B) \leq \deg(A)$. Since $\deg(A) \leq g-1 = p_a(C)$, we have $h^1(C,B) \neq 0$. Hence there is an integer r with $B \cong R^{\otimes r}$ (Proposition 2.3). We have $\deg(B) = 2r$ and $h^0(C,B) = r+1$. Since $p_a(Y) - p_a(C) = 1$, we have $\deg(B) \leq \deg(w_*(B)) \leq \deg(B)+1$ and $h^0(C,B) = h^0(Y,w_*(B)) \leq h^0(Y,A)+1$. Since Y is Gorensein but not hyperelliptic and $h^1(Y,A) \neq 0$, we have $\deg(A) > 2(h^0(Y,A)-1)$ ([EKS], Th. A of the Appendix with J. Harris). We distinguish the following two cases.

(a) Assume $\deg(A) = 2r+1$. Hence we must have $A \cong w_*(B)$. Thus $h^0(Y, A) = r+1$. Since Y is not hyperelliptic, the sheaf A must be spanned.

(b) Assume $\deg(A) = 2r$. Hence we have $h^0(Y, A) = r$. Let τ: $X \to C$ be the birational morphism such that $\pi = w \circ \tau$. Since R and B are locally free, we have $\deg(\tau^*(B)) = \deg(B)$ and $\tau^*(B) = \pi^*(A)/\operatorname{Tors}(\pi^*(A))$. Hence $\deg(\pi^*(A)/\operatorname{Tors}(\pi^*(A))) = \deg(A)$. By [EKS], Lemma 1 of the Appendix with J. Harris, A is a line bundle. In particular $w^*(A)$ has no torsion and hence $B \cong w^*(A)$. Since A is spanned and the pair $(B, H^0(C, B))$ induces a two to one morphism γ, the pair $(A, H^0(Y, A))$ induces a morphism μ: $Y \to \mathbf{P}^{r-1}$ with $\deg(\mu) > 1$. Since A is spanned, we have $\deg(A) = \deg(\mu) \cdot \deg(\mu(Y))$. Since $\deg(\mu(Y)) \geq r-1$, we obtain $\deg(\mu) = 2$ and $\deg(\mu(Y)) = r-1$. Hence either $r = 2$ or $\deg(\mu) = 2$, $\deg(\mu(Y)) = r-1$ and $\mu(Y)$ is either a linearly normal elliptic curve or a possibly singular rational curve with $p_a(\mu(Y)) \leq 1$. The curve $\mu(Y)$ cannot be a smooth rational curve because the map μ is induced by a complete linear system. The curve $\mu(Y)$ cannot be elliptic, because an elliptic curve cannot be the target (through a linear projection) of the smooth rational curve $\gamma(C) \subset \mathbf{P}^{r-1}$. Hence we are in the set-up of Example 2.7.

Remark 2.9: Notice that 2.7 and 2.8 give a way to construct all such spanned special sheaves A, since the hyperelliptic curve C is uniquely determined by Y.

Now we study special linear systems on trigonal Gorenstein curves with trigonal pencil locally free, i.e. with a degree 3 morphism $Y \to \mathbf{P}^1$. First, we will consider the case of spanned line bundles.

THEOREM 2.10: *Let Y be an integral Gorenstein projective curve with $g := p_a(Y) \geq 5$ and $L \in \operatorname{Pic}^3(Y)$ with $h^0(Y, L) = 2$. Assume Y not hyperelliptic. Let A be a spanned line bundle on Y with $0 < x := \deg(A) \leq g-1$. Set $r := h^0(Y, A) - 1 \geq 2$. Then either $A \cong L^{\otimes r}$ and in particular $x = 3r$ or there is an effective Cartier divisor U on Y such that U is the base locus of $|\omega_Y - (g-x+r-1)L|$ and $\omega_Y \cong A \otimes \mathbf{O}_Y(U) \otimes L^{\otimes(g-x+r-1)}$.*

Proof: Let m be the Maroni invariant of the pair (Y, L) given by [RS], Th. 3.6; as in the smooth case m is the unique integer such that $h^0(Y, L^{\otimes i}) = i+1$ if $0 \leq i < g-m$, $h^0(Y, L^{\otimes i}) = g-m-1+2(i-g-m+1) = 2i-g+m+1$ if $g-m \leq i < m+2$ and $h^0(Y, L^{\otimes i}) = 3i+1-g$ if $i \geq m+2$ ([RS], Cor. 2.5). Hence m is an integer with $(g-4)/3 \leq m \leq (g-2)/2$. Since Y is not hyperelliptic, L is spanned. Since Y is Gorenstein and not hyperelliptic, the canonical map of Y is an embedding ([Ro], Th. 15) and we will see Y as a linearly normal curve of $\mathbf{P}^{g-1}$ with $\deg(Y) = 2g-2$. By [RS], Y is contained in a two-dimensional smooth rational scroll $S \subset \mathbf{P}^{g-2}$ with $\deg(S) = g-2$ and

$$S \cong \mathbf{P}(\mathbf{O}_{\mathbf{P}^1}(m) \otimes \mathbf{O}_{\mathbf{P}^1}(g-2-m)).$$

Hence S is isomorphic to the Hirzebruch surface F_e with $e = g-2-2m$ and we will take as base of $\mathrm{Pic}(F_e) \cong \mathbf{Z}^{\otimes 2}$ a ruling R and a curve, E, with minimal self-intersection, i.e. R is a line of $\mathbf{P}^{g-1}$, $E^2 = -e$, $E \cdot R = 1$ and $R^2 = 0$. We have $K_S = -2E-(e+2)R$, $\mathbf{O}_S(1) = E+(g-2-m)R$, $Y = 3E+(m+2)R$, $\omega_Y = (K_S+Y)|Y = (E+(m-e)R)|Y$ (see e.g. [MS], pp. 172–173). We will follow the proof of the smooth case given in [MS], Prop. 1. The restriction map ρ: $H^0(S, K_S+Y) \to H^0(Y, \omega_Y)$ is bijective because $h^1(S, K_S) = h^2(S, K_S) = 0$. Notice that the linear systems $|A|$ and $|\omega_Y - A|$ are not empty. Fix a general $D \in |A|$ and a general $D' \in |\omega_Y - A|$. Since A and $\omega_Y - A$ are locally free, the divisor $D+D'$ is defined ([C]) and $D+D' \in |\omega_Y|$; here we use only that the tensor product of two spanned line bundles is spanned and that $M \otimes M^* \cong \mathbf{O}_Y$ for every $M \in \mathrm{Pic}(Y)$. Since ρ is bijective, we have $r+1 = h^0(S, \mathbf{I}_{D'} \otimes (K_S+Y))$. Since $\deg(S) = g-2 = (K_S+Y)^2$ and $\deg(D') = 2g-2-x \geq g-1$, the linear system $\mathbf{P}(H^0(S, \mathbf{I}_{D'} \otimes (K_S+Y)))$ on S has a base component, T. Call Z a general divisor of the moving part of $\mathbf{P}(H^0(S, \mathbf{I}_{D'} \otimes (K_S+Y)))$ and $\{Z\}$ the corresponding (perhaps non-complete) linear system. Hence $\dim(\{Z\}) = r$ and Z is nef. If $T \in |E+yR|$, $y \geq 0$, $\{Z\}$ is a subseries of $(m-e-y)R$. We have $m-e-y < g-m$. Hence in this case A is obtained from $L^{\otimes r}$ adding an effective Cartier divisor of degree $x-3r$; since $h^0(Y, L^{\otimes r}) = r+1$ and A is assumed to be spanned, we have $A \cong L^{\otimes r}$ and $x = 3r$. Since $K_S+Y = E+(m-e)R = E+(3m-g+2)R$, it remains the case $T \in |yR|$ and $Z \in |E+(g-2-m-y)R|$ with $0 < y \leq g-2-m-e = m$; here we use that Z is nef and hence $Z \cdot E \geq 0$. Since $\dim(\{Z\}) = r$, we have $2+2(g-2-m-y)-e \geq r+1$, i.e. $g-2y \geq r+1$. We call Z a sufficiently general element of $\{Z\}$ (remember that for fixed D' we may still take D general). Hence Z is a smooth rational curve of degree $Z \cdot H = g-2-y$. Since A is spanned, D' contains the scheme-theoretic intersection $T \cap Y$ and $D \subset Z$. We have $2g-2-x = \deg(D') \leq yR \cdot Y + Z^2 = g-2-y$, i.e. $x \geq g-y$. If D is contained in a hyperplane of Z, then $x \leq \deg(Z) = g-2-y$, contradiction. Hence we have $\langle D \rangle = \langle Z \rangle$. By the geometric form of Riemann–Roch we obtain $r = x-g+y+1$, i.e. $y = g-x+r-1$. Thus $D \in |\omega_Y - (g-x+r-1)R - U|$ with U non-negative Cartier divisor.

With the terminology of [MS], p. 173, if A is as in the second case of the statement of 2.10, then $A \in V_n^r$. As in the smooth case (see [MS], Cor. 2) from 2.10 we obtain the following result.

COROLLARY 2.11: *Let Y be an integral Gorenstein projective curve with $g := p_a(Y) \geq 5$ and $L \in \mathrm{Pic}^3(Y)$ with $h^0(Y,L) = 2$. Assume Y not hyperelliptic. Let A be a spanned line bundle on Y with $0 < x := \deg(A) \leq g-1$. Set $r := h^0(Y,A) - 1 \geq 2$. Then $x \geq 3r$ and $x = 3r$ if and only if either $A \cong L^{\otimes r}$ or $x = g-1$ and $A \cong \omega_Y \otimes L^{*(g-1)/3}$.*

Now, at least if Y has only ordinary nodes or ordinary cusps as singularities, we will reduce the case of an arbitrary spanned rank 1 torsion free sheaf A to the case of a spanned line bundle, A', on the partial normalization, C, of Y in which we normalize only the subset $\mathrm{Sing}(A)$ of $\mathrm{Sing}(Y)$. C is a trigonal curve but its Maroni invariant does not depend only on the Maroni invariant of Y but also on the "position" of the set $\mathrm{Sing}(A)$ in the rational scroll containing the canonical image of Y. To make this assertion more explicit we need the following definition.

Definition 2.12: Let $S \subset \mathbf{P}^{g-1}$, $g \geq 5$, be a minimal degree surface which is not a cone over a rational normal curve of $\mathbf{P}^{g-2}$. Hence $\deg(S) = g-2$ and there exists an integer m with $(g-2)/2 \leq m < g-2$ wuch that $S \cong \mathbf{P}(\mathbf{O}_{\mathbf{P}^1}(m) \otimes \mathbf{O}_{\mathbf{P}^1}(g-2-m))$. The integer m is unique and we will call it the Maroni invariant of S. Take a finite subset B of S with $0 < b := \mathrm{card}(B) \leq g-5$. Fix $P \in B$ and let $S_1 \subset \mathbf{P}^{g-2}$ be the image of the surface S from the projection from P. Since S is not a cone, S_1 is a minimal degree surface of $\mathbf{P}^{g-2}$ and it is not a cone, unless $m = g-3$ and P is contained in the unique section of the ruling of S with negative self-intersection. We assume that this is not the case. Hence $S_1 \cong \mathbf{P}(\mathbf{O}_{\mathbf{P}^1}(m_1) \otimes \mathbf{O}_{\mathbf{P}^1}(g-3-m_1))$. The Maroni invariant m_1 of S_1 is either m or $m-1$. If $b = 1$ we stop. Assume $b > 1$. The image, B_1, of $B \setminus \{P\}$ through the projection from P is a subset of S_1 with $\mathrm{card}(B_1) = b-1$. Hence we may apply the same construction to the pair (S_1, B_1). We will say that B is good for S if we may repeat the construction b times without ever finding a cone. We will call the Maroni invariant of the last surface $S_b \subset \mathbf{P}^{g-1-b}$ the Maroni invariant of the pair (S,B) (or just of B if there is no danger of misunderstanding) and we will denote it by $m_S(B)$ (or just $m(B)$ if there is no danger of misunderstanding).

Remark 2.13: Take S and B as in Definition 2.12. If B is general in S, then B is good and $m(B) = \max\{m-b, [(g-1-b)/2]\}$.

Remark 2.14: Let R be the completion of the local ring of a curve at a point which is either an ordinary node or an ordinary cusp. Assume $\mathrm{char}(\mathbf{K}) \neq 2$ if R is the completion of an ordinary node and $\mathrm{char}(\mathbf{K}) \neq 2,3$ if R is the completion of an ordinary cusp. Let $\mathbf{m}$ be the maximal ideal of R. Then every rank 1 torsion free module over R is isomorphic either to R or to $\mathbf{m}$ (see [D'S] or [Se], Prop. 3

at p. 163, for the nodes, [Co], p. 24, for nodes and cusps if $\mathrm{char}(\mathbf{K}) = 0$); to avoid any misunderstanding with the notation of [Se], bottom of p. 165, we stress that in the nodal case as torsion free modules we allow only modules with rank 1 on both components of $\mathrm{Spec}(R)$. Now fix $P \in \mathrm{Sing}(Y)$ with P an ordinary node or an ordinary cusp and a rank 1 torsion free sheaf A on Y with $P \in \mathrm{Sing}(A)$. Hence A is formally isomorphic to the maximal ideal of the local ring of P in Y. Let u: $C \to Y$ be partial normalization of Y at P. Hence $p_a(C) = p_a(Y) - 1$. Set $A' := u^*(A)/\mathrm{Tors}(u^*(A))$. Notice that the maximal ideal of the local ring of P in Y is formally isomorphic to the germ at P of $u_*(\mathbf{O}_C)$. Hence $A \cong u_*(A')$. Since C is smooth at each point of $u^{-1}(P)$, A' is a rank 1 torsion free sheaf on C with A' smooth along $u^{-1}(P)$, $\mathrm{card}(\mathrm{Sing}(A')) = \mathrm{card}(\mathrm{Sing}(A)) - 1$ and $\deg(A') = \deg(A)-1$ ([Co], p. 18, or the proof of [EKS], Lemma 1 of the Appendix with J. Harris). The last equality follows also from the definition of degree because $A \cong u_*(A')$.

THEOREM 2.15: *Assume* $\mathrm{char}(\mathbf{K}) \neq 2,3$. *Let Y be an integral Gorenstein projective curve with $g := p_a(Y) \geq 6$ and $L \in \mathrm{Pic}^3(Y)$ with $h^0(Y,L) = 2$. Assume Y not hyperelliptic. Let A be a spanned rank 1 torsion free sheaf on Y. Set $B := \mathrm{Sing}(A)$, $b := \mathrm{card}(\mathrm{Sing}(A))$, $d := \deg(A)$. Assume $0 < d \leq g-1$, $0 < b \leq g-5$ and that Y has only ordinary nodes or ordinary cusps at each point of $\mathrm{Sing}(A)$. Let u: $C \to Y$ be partial normalization of Y at the points of $\mathrm{Sing}(A)$; hence $p_a(C) = g - b \geq 5$. Set $A' := u^*(A)/\mathrm{Tors}(u^*(A))$. We have $A' \in \mathrm{Pic}^{d-b}(C)$. We have $h^0(C,A') \geq h^0(Y,A) = r+1$ (Lemma 2.1). Since $u^*(L)$ induces a degree 3 pencil on C, C is trigonal. Let $S \subset \mathbf{P}^{g-1}$ be the rational normal scroll associated to S. Then B is good for S in the sense of Definition 2.12 and the Maroni invariant of C is the Maroni invariant $m(B)$ of the pair (S,B). A' is classified by Theorem 2.10 and $A \cong u_*(A')$.*

Proof: Let v: $S' \to S$ be the blowing-up of S at each point of $\mathrm{Sing}(A)$ and α': $S \to \mathbf{P}^1$ the morphism induced by the ruling α: $S \to \mathbf{P}^1$ which induces L. Notice that each fiber of the ruling α contains at most one point of $\mathrm{Sing}(A)$. Hence each fiber of α' is either a smooth rational curve or the union of two smooth rational curves with self-intersection 1 and one of them has intersection multiplicity at most one with C. Hence we may blow-down each of the components of the reducible fibers of α' which are mapped to curves in S obtaining a minimal ruled surface α'': $S'' \to \mathbf{P}^1$ containing C. We claim that $S'' \subset \mathbf{P}^{g-1-b}$ is obtained from $S \subset \mathbf{P}^{g-1}$ by the projection from the b points of $\mathrm{Sing}(A)$ and $C \subset S'' \subset \mathbf{P}^{g-1-b}$ is obtained from Y in the same way. To check the claim we need to check that $\dim(\langle \mathrm{Sing}(A) \rangle) = b-1$ and that for every length 2 subscheme,

τ, of $Y \setminus \mathrm{Sing}(A)$ we have $\dim(\langle \mathrm{Sing}(A) \cup \tau \rangle) = b+1$. Indeed the last equality for every τ would be equivalent to the assertion that the rational map from Y to C obtained projecting from the set $\mathrm{Sing}(A)$ is the inverse of the partial normalization u. Fix $P \in \mathrm{Sing}(A)$ and consider the curve $Y' \subset \mathbf{P}^{g-2}$ obtained projecting Y from P. Let Y'' be the partial normalization of Y at P. Since P is an ordinary node or an ordinary cusp of Y, we have $p_a(Y'') = g-1$. Y'' cannot be hyperelliptic because the trigonal pencil of Y is locally free. The rational map $Y \to Y'$ obtained projecting from P cannot have degree at least two because its image would be a rational normal curve, its degree would be two and hence Y'' would be hyperelliptic, contradiction. Since Y has multiplicity two at P and the rational map $Y \to Y'$ obtained projecting from P is birational, we have $\deg(Y') = 2g-4$. Since $p_a(Y) \geq g-1$, we obtain easily in arbitrary characteristic that $Y' \cong Y''$ and that Y' is canonically embedded in $\mathbf{P}^{g-2}$. Iterating the projection $b-1$ times we obtain the claim. Call $m(A)$ the Maroni invariant of S'', i.e. the integer such that $S'' \cong \mathbf{P}(\mathbf{O}_{\mathbf{P}^1}(m(A)) \otimes \mathbf{O}_{\mathbf{P}^1}(g-2-b-m(A)))$. We may apply 2.8 to A'. The obvious isomorphism between A and $u_*(A')$ was claimed in 2.14.

COROLLARY 2.16: *Assume* $\mathrm{char}(\mathbf{K}) \neq 2,3$. *Let Y be an integral projective curve with $g := p_a(Y) \geq 5$ and $L \in \mathrm{Pic}^3(Y)$ with $h^0(Y,L) = 2$. Assume Y not hyperelliptic and that Y has only ordinary nodes or ordinary cusps as singularities. Let A be a rank 1 spanned torsion free sheaf on Y. Set $d := \deg(A)$ and $r := h^0(Y,A) - 1$. Assume $d \leq g-1$. Then $d \geq 3r$ and $d = 3r$ if and only if $A \cong L^{\otimes(d/3)}$ or $d = g-1$ and $A \cong \omega_Y \otimes L^{*(g-1)/3}$.*

It seems useful to consider the following concept; essentially, it is the reason why 2.15 and 2.16 work for curves with ordinary nodes and ordinary cusps.

Definition 2.17: Let A be a rank 1 torsion free sheaf on Y. Set $\delta - \deg(A) := \deg(\pi^*(A)/\mathrm{Tors}(\pi^*(A)))$. The integer $\delta - \deg(A)$ will be called the δ-degree of A.

Remark 2.18: Let A be a rank 1 torsion free sheaf on Y. We have $\delta - \deg(A) \leq \deg(A)$ and $\delta - \deg(A) = \deg(A)$ if and only if A is locally free ([EKS], Lemma 1 of the Appendix). Furthermore, $\delta - \deg(A) \leq \deg(A) - \mathrm{card}(\mathrm{Sing}(A))$. Let $u: C \to Y$ be partial normalization of Y in which we normalize only the points of $\mathrm{Sing}(A)$. Then $\delta - \deg(A) = \deg(u^*(A)/\mathrm{Tors}(u^*(A)))$. If $\mathrm{char}(\mathbf{K}) \neq 2,3$ and Y has only ordinary nodes or ordinary cusps at every point of $\mathrm{Sing}(A)$, then $\delta - \deg(A) = \deg(A) - \mathrm{card}(\mathrm{Sing}(A))$ (Remark 2.14).

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